The Association
of
Engineering and Shipbuilding Draughtsmen.

THE STRENGTH OF ROTATING DISCS.

WITH SPECIAL REFERENCE TO STEAM TURBINES.

By R. GARDNER (Member).

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LIST OF SYMBOLS.

t = thickness (inches) of disc at any radius r (inches).

p = hoop (tensile) stress (lbs. per sq. in.) at any radius r (inches).

q = radial (tensile) stress (lbs. per sq. in.) at any radius r (inches).

R = neck radius of disc (inches).

R_0 = bore ,, ,, ,,

R_1 = mean radius of rim or drum (inches).

R_2 = ,, ,, blading (inches).

T = neck thickness (inches) of disc (at radius R inches).

T_0 = central ,, ,, ,, disc of uniform strength.

J = nett radial section of rim (sq. inches) or mean thickness of drum (inches).

W_1 = weight of rim (lbs.), or weight of drum (lbs. per inch length of drum).

W_2 = total weight (lbs.) of blading on a disc or weight of blading (lbs. per inch length of drum).

P = hoop (tensile) stress (lbs. per sq. inch) at radius R (inches).

Q = radial (tensile) stress (lbs. per sq. inch) at radius R (inches).

P_0 = hoop (tensile) stress (lbs. per sq. inch) at radius R_0 (inches) of a bored disc or at centre of a solid flat disc.

F = hoop (tensile) stress (lbs. per sq. inch) in rim or in drum.

$M = R \div R_0$.

n = (constant) index for bored discs.

C = constant for a bored disc $= tr^n$

A and B are constants for determination of P. See Figs. 3 and 4.

A_0 and B_0 are constants for determination of P_0. See Figs. 5 and 6.

U, V, and S are dimensions (inches) used in calculating flank stress. See Fig. 13.

λ = centrifugal load (lbs. per inch of circumference of radius S).

D and D_0 are dimensions (inches) used in calculating shrinkage stress. See Fig. 14.

$\mu = D_0 \div D$.

δ = shrinkage or force fitting strain (inches per inch). This symbol is also prefixed to other quantities in which case it means "increment."

N = revs. per min.

$K = 8 \times (N \div 1000)^2$.

$\gamma = K \div (2\ Q)$.

g = gravitational acceleration (inches per sec.^{2})
E = Young's Modulus of Elasticity (lbs. per sq. inch)
σ = Poisson's Ratio
e = base of natural or Naperian logarithms

$\left.\right\}$ See Appendix.

Y_1 Y_2 Z α_1 α_2 β_1 β_2 and ω are temporary constants used in deriving the formulæ.

INTRODUCTION.

In treating of the strength of rotating discs for steam turbines, it is usual to begin by lamenting that mathematicians have no *exact* way of calculating the stresses in discs of the forms actually used. Unfortunately the stresses in any part of any machine or structure cannot be calculated exactly; it is always a matter of approximation.

The usual form of disc, which has its thickness inversely proportional to a power of the radius, is eminently suitable for practical purposes. It is not uniformly stressed throughout—the hoop stress is usually greater near the hub than near the rim. We are, however, generally limited by the requirement of a minimum thickness at the rim, and therefore, if we made the hoop stress uniform, the disc thickness at the hub would be inconveniently great.

Outlines of discs of the type referred to are shown in Fig. 1. The thickness at the rim is the same in all cases, and the indices (i.e., the powers of the radius to which the thickness is inversely proportional) are respectively $\frac{1}{2}$, 1, 3/2, and 2. Of course, the shape of the disc is actually formed by straight cuts, but it is a fair assumption that only small deviations from theoretical stress are caused by small modifications of theoretical thickness.

There is no great difficulty in following the mathematical theory, nothing beyond a fairly elementary knowledge of differential equations being involved, and I propose, in these notes, to give in some detail the working out for the general case. It is sometimes asked: Cannot the stresses be found by some graphical step by step process like those in a beam irregularly loaded? The answer is difficult, for the radial and hoop stresses are both bound up with the radial and hoop strains and with each other.

THE STRENGTH OF ROTATING DISCS.

With Special Reference to Steam Turbines.

BY R. GARDNER (Member).

1. THE STRAIN EQUATIONS.

p = hoop stress at radius r inches.
q = radial stress at radius r inches.
E = Young's Modulus of elasticity.
σ = Poisson's Ratio.

Considering a point in the disc at radius r which increases by strain to $r+u$, we have, as shewn in Morley's "Strength of Materials," Par. 126—

$$\text{Hoop Strain} = \frac{u}{r} = \frac{p-\sigma q}{E} \qquad \text{.........(1)}$$

$$\text{Radial Strain} = \frac{du}{dr} = \frac{q-\sigma p}{E} \qquad \text{.........(2)}$$

Putting (1) in the form—

$$Eu = pr - \sigma qr$$

and differentiating, we obtain :—

$$Edu = pdr + rdp - \sigma\, qdr - \sigma\, rdq.$$

From (2) we have—

$$Edu = qdr - \sigma\, pdr$$

Therefore—

$$pdr + rdp - \sigma\, qdr - \sigma\, rdq = qdr - \sigma\, pdr$$

$$\text{or, } rdp + (1+\sigma)\, pdr = \sigma\, rdq + (1+\sigma)qdr \qquad \text{.........................(3)}$$

We thus have a relation, not involving strain, between the hoop and radial stresses.

Let both sides of equation (3) be multiplied by r^{σ}. The left hand side is then the differential of $pr^{1+\sigma}$, or $d\,(pr^{1+\sigma})$, therefore, making this substitution—

$$d\,(pr^{1+\sigma}) = \sigma\, r^{1+\sigma}\, dq + (1+\sigma)\, qr^{\sigma}\, dr \qquad \text{.........................(4)}$$

2. THE EQUILIBRIUM CONDITION.

We now proceed to find, from considerations of equilibrium, a further equation connecting p and q so that they may be separately determined.

The centrifugal force on 1 cub. inch of steel (due to its own mass) is

$$\frac{\cdot 28}{g} \times r \times \left(\frac{2\pi N}{60}\right)^2 \text{ lbs.}$$

N = revs. per min.

g = 12 × 32.2.

(1 cub. inch of steel weighs about .28 lb.)

The expression reduces to

$$8 \times \left(\frac{N}{1000}\right)^2 \times r$$

or if we put

$$K = 8 \times \left(\frac{N}{1000}\right)^2$$

we have centrifugal force = Kr lbs. per cub. inch of steel.

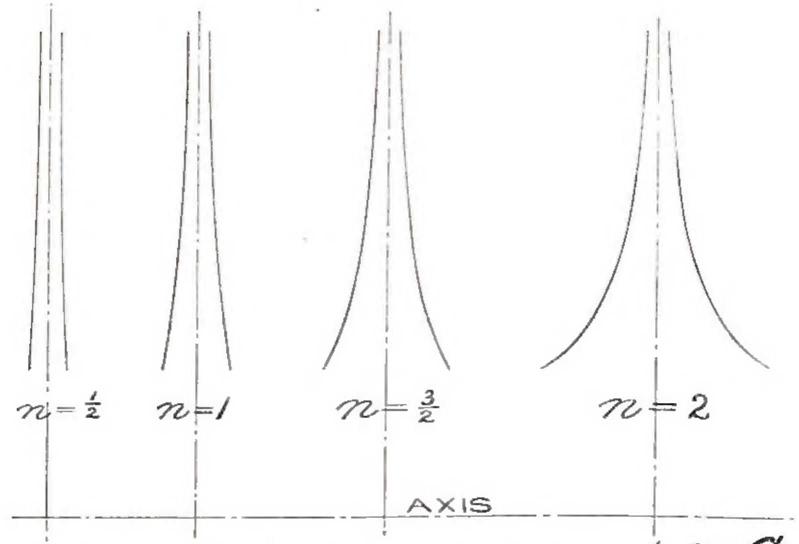
Now consider an element of the disc as indicated in Fig. 2. The thickness of the disc at this point is t inches, and the element is that part of the sector $d\theta$ between radii r and r + dr. The volume of the element is $rd\theta \times t \times dr$ cub. inches. t is, of course, a function of r, but its variation does not enter into the present discussion; an attempt to allow for the variation of t over the range dr would give rise to a term involving $d\theta \times dt \times dr$, and this, being an infinitesimal of the third order, is negligible, since we are dealing with terms of the second order.

Multiplying the volume of the element by Kr, we have centrifugal force on element due to its own mass =

$$Kr^2 \ t \ dr \ d\theta$$

The radial force on the *inner* face of the element

$$= \text{stress} \times \text{area}$$
$$= q \times t \ r \ d\theta$$



SECTIONS OF DISCS OF THE FORM $tr^n = C$

FIG. 1.

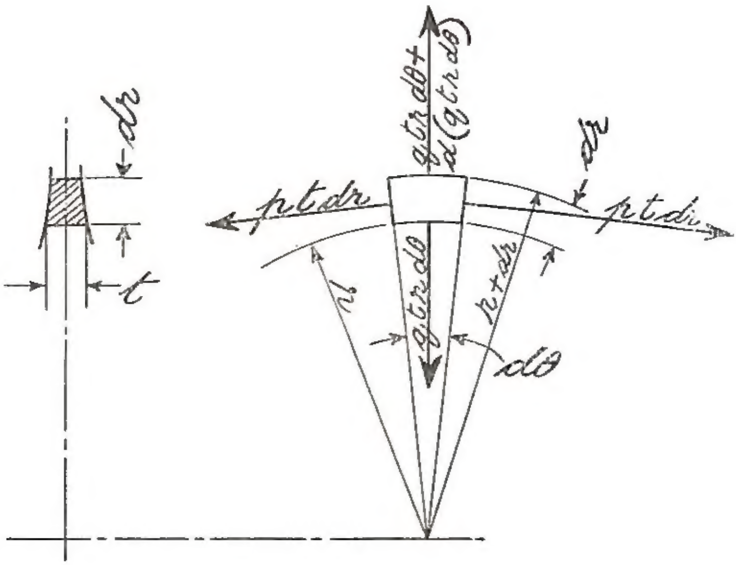


DIAGRAM OF FORCES ON ELEMENT OF DISC.

FIG. 2.

Similarly, the radial force on the *outer* face

$$= (q + dq) \times (t + dt)\ (r + dr)\ d\theta$$
$$= (qtr + qt\ dr + tr\ dq + rq\ dt)\ d\theta$$

terms containing infinitesimals of higher order than the second being neglected.

Subtracting the force on the inner face from the force on the outer face, we have nett outward radial force

$$= (qt\ dr + tr\ dq + rq\ dt)\ d\theta$$
$$= d\ (qtr)\ d\theta$$

The hoop tension on *each* end of the element

$$= \text{stress} \times \text{area}$$
$$= p \times t\ dr$$

and the angle between its direction and the tangent at the centre of the element is $\frac{d\theta}{2}$. Hence the inward radial force due to the hoop tensions on *both* ends of the element

$$= 2 \times pt\ dr\ \sin \frac{d\theta}{2}$$
$$= pt\ dr\ d\theta$$

We have then the equation of equilibrium :

$$pt\ dr\ d\theta = Kr^2\ t\ dr\ d\theta + d\ (qtr)\ d\theta$$
$$\text{or,}\ (p - Kr^2)\ t\ dr = d\ (qrt) \qquad \text{.....................(1)}$$

3. A THIRD DATUM NECESSARY.

In the process of determining the stresses p and q we have obtained two equations, viz., (3) of Par. 1, and (1) of Par. 2. In the latter, however, a third variable, t, is involved, hence a third condition must be specified before a solution is possible.

In the case of a solid disc, (i.e., a disc not bored through the centre), we may make the very simple condition—

$$p = q = \text{constant}$$

It will be seen that this satisfies the strain equation, (3) of Par. 1, and the solution is readily completed by inserting the condition in the equation of equilibrium, (1) of Par. 2.

We shall return later to consideration of this simple form of disc (see Par. 17). It is only in special cases that a disc or rotor is solid. When the disc is centrally bored, we cannot secure the condition $p = q$ everywhere, for at the bore $q = 0$ and p does not $= 0$, and it becomes necessary to adopt a formal method of working out the problem.

4. SHAPE OF CENTRALLY BORED DISC.

The form of disc we adopt for central bore has its thickness specified by the relation—

$$tr^{n} = \text{constant} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

n being a constant for the given disc.

Differentiating (1),

$$r^{n}\,dt + nr^{n-1}t\,dr = 0,$$

whence our third datum may be written in the form—

$$n = -\frac{r}{t}.\frac{dt}{dr} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

5. DETERMINATION OF THE STRESSES.

Referring to Eq. (1) of Par. 2, the right hand side

$$d\,(qrt) = qr\,dt + qt\,dr + rt\,dq.$$

The equation is therefore

$$p - Kr^2 = q.\frac{r}{t}.\frac{dt}{dr} + q + r.\frac{dq}{dr}$$

Therefore, from Eq. (2) of Par. 1—

$$p - Kr^2 = -nq + q + r\frac{dq}{dr}$$

$$\text{or} \quad r\frac{dq}{dr} - (n-1)\,q - p = -Kr^2 \quad \ldots\ldots\ldots(1)$$

Multiplying by $r^{1+\sigma}$:

$$r^{2+\sigma}\frac{dq}{dr} - (n-1)\,qr^{1+\sigma} - pr^{1+\sigma} = -Kr^{3+\sigma}.$$

Differentiating, and for $d\,(pr^{1+\sigma})$ substituting the right hand side of Eq. (1), Par. 1—

$$\begin{aligned} r^{2+\sigma}\frac{d^2q}{dr^2} &+ (2+\sigma)\,r^{1+\sigma}\frac{dq}{dr} \\ &- (n-1)\,r^{1+\sigma}\frac{dq}{dr} - (n-1)\,(1+\sigma)\,qr^{\sigma} \\ &- \sigma r^{1+\sigma}\frac{dq}{dr} - (1+\sigma)\,qr^{\sigma} \\ &= -(3+\sigma)\,Kr^{2+\sigma}. \end{aligned}$$

Collecting the terms; and dividing throughout by r^{σ}:

$$r^2\frac{d^2q}{dr^2} + (3-n)\,r\frac{dq}{dr} - n\,(1+\sigma)\,q = -(3+\sigma)\,Kr^2 \ldots\ldots(2)$$

This is a well-known type of differential equation of which the solution is—

$$q = Y_1\,r^{\alpha_1} + Y_2\,r^{\alpha_2} + Z\,r^2$$

where Y_1, Y_2, Z, α_1, α_2 are constants. α_1, α_2 are found by substituting

$$q = \text{constant} \times r^{\alpha}$$

in the left hand side of (2) and equating it to zero. This gives—

$$\alpha\ (\alpha-1)+(3-n)\ \alpha-n\ (1+\sigma)=0$$

or—

$$\alpha^2+(2-n)\ \alpha-n\ (1+\sigma)=0$$

and α_1, α_2 are the roots of this equation :

$$\alpha_1 = -1 + \frac{n}{2} + \omega$$

$$\alpha_2 = -1 + \frac{n}{2} - \omega$$

where $$\omega = \sqrt{1 + \sigma n + \left(\frac{n}{2}\right)^2}$$

The constant Z is found by putting

$$q = Zr^2$$

in (2). The result, after dividing throughout by r^2, is—

$$2\ Z+2\ (3-n)\ Z-n\ (1+\sigma)\ Z=-(3+\sigma)\ K$$

whence—

$$Z = -\frac{(3+\sigma)\ K}{8-(3+\sigma)\ n}$$

The solution of (2) now stands in the form

$$q = Y_1 r^{\alpha_1} + Y_2 r^{\alpha_2} - \frac{(3+\sigma)\ K}{8-(3+\sigma)\ n}.\ r^2 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ (3)$$

To find the hoop stress, we have from (1)—

$$p = Kr^2 + r\frac{dq}{dr} - (n-1)\ q$$

and substituting for q from (3)—

$$p = Kr^2 + \alpha_1 Y_1 r^{\alpha_1} + Y_2 \alpha_2 r^{\alpha_2} + 2\,Zr^2$$
$$-(n-1)Y_1 r^{\alpha_1} - (n-1)\,Y_2 r^{\alpha_2} - (n-1)Zr^2$$
$$= (1+\alpha_1-n)Y_1 r^{\alpha_1} + (1+\alpha_2-n)Y_2 r^{\alpha_2}$$
$$+ [K+(3-n)Z]\,r^2$$

$$\text{And } K+(3-n)\,Z = K - \frac{(3-n)\,(3+\sigma)}{8-(3+\sigma)\,n}.\,K$$
$$= \frac{8-(3+\sigma)\,n-(3-n)\,(3+\sigma)}{8-(3+\sigma)\,n}.\,K$$
$$= -\frac{(1+3\sigma)\,K}{8-(3+\sigma)n}$$

So that—

$$p = \beta_1 Y_1 r^{\alpha_1} + \beta_2 Y_2 r^{\alpha_2} - \frac{(1+3\sigma)\,K}{8-(3+\sigma)n}.\,r^2 \quad \ldots\ldots(4)$$

where—

$$\beta_1 = 1+\alpha_1-n = -\frac{n}{2} + \omega$$
$$\beta_2 = 1+\alpha_2-n = -\frac{n}{2} - \omega$$

The remaining constants Y_1, Y_2 may be determined from the fact that q=O at the bore radius R_0 and by assigning to q at the outer radius R of the disc the value Q.

6. EVALUATION OF THE CONSTANTS.

With these data, we have from Eq. (3) of Par. 5—

$$Y_1 R^{\alpha_1} + Y_2 R^{\alpha_2} = \frac{3+\sigma}{8-(3+\sigma)n}.\,KR^2+Q$$

$$Y_1 R_0^{\alpha_1} + Y_2 R_0^{\alpha_2} = \frac{3+\sigma}{8-(3+\sigma)n}.\,KR_0^2$$

The solution of this pair of simultaneous equations in Y_1, Y_2 is a question of elementary algebra.

The solution is—

$$Y_1 = \frac{3+\sigma}{8-(3+\sigma)n} \cdot \frac{R_0^{\alpha_2} - R_0^2 R^{\alpha_2 - 2}}{R^{\alpha_1} R_0^{\alpha_2} - R_0^{\alpha_1} R^{\alpha_2}} \cdot KR^2$$

$$+ \frac{R_0^{\alpha_2}}{R^{\alpha_1} R_0^{\alpha_2} - R_0^{\alpha_1} R^{\alpha_2}} \cdot Q$$

$$Y_2 = -\frac{3+\sigma}{8-(3+\sigma)n} \cdot \frac{R_0^{\alpha_1} - R_0^2 R^{\alpha_1 - 2}}{R^{\alpha_1} R_0^{\alpha_2} - R_0^{\alpha_1} R^{\alpha_2}} \cdot KR^2$$

$$- \frac{R_0^{\alpha_1}}{R^{\alpha_1} R_0^{\alpha_2} - R_0^{\alpha_1} R^{\alpha_2}} \cdot Q$$

The stresses q and p may now be calculated from the formulæ (3) and (4) of Par. 5. Such calculation would obviously be a tedious matter, and it is much easier to use curves like those given by H. M. Martin by means of which the stresses at any radius may be found. Moreover, in all actual cases of highly stressed discs, the greatest stress is the hoop stress at the bore. If we confine ourselves to the calculation of the stresses at the inner and outer radii of the disc, the construction of curves becomes simplified.

7. PRACTICAL SHAPE OF THE STRESS FORMULÆ.

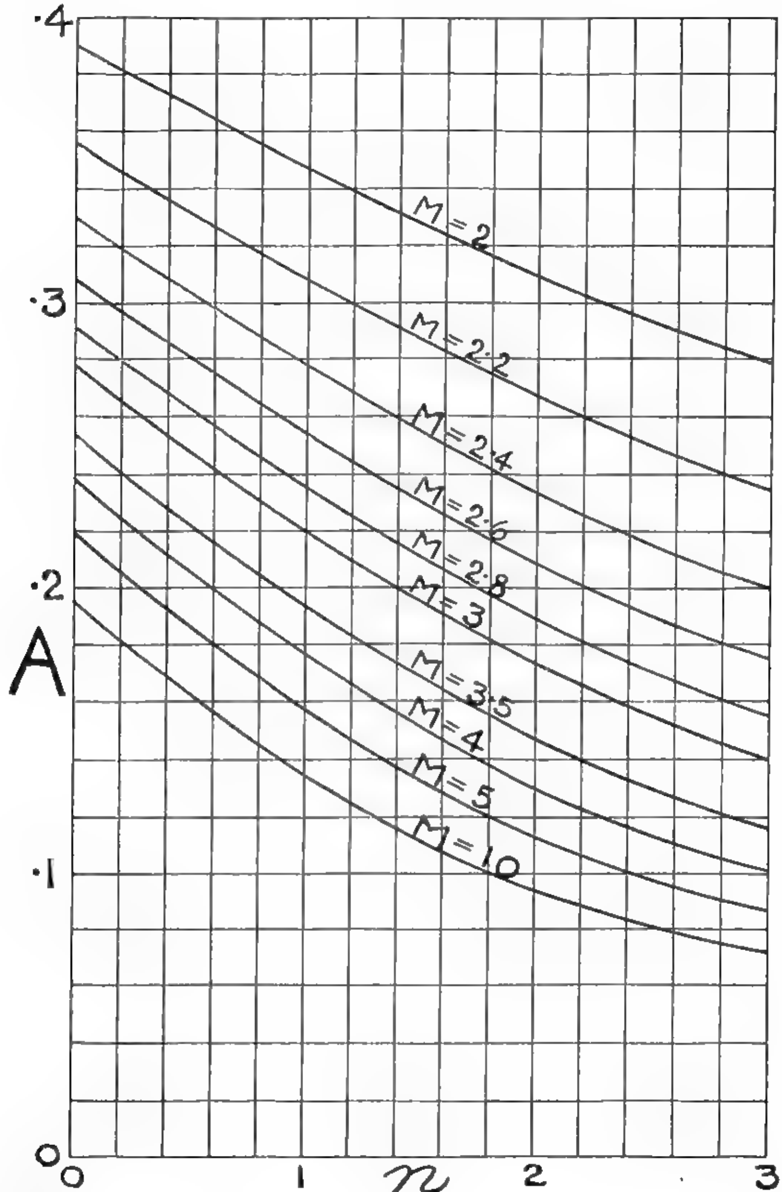
Let $p = P_0$ at $r = R_0$

and $p = P$ at $r = R$.

Let also

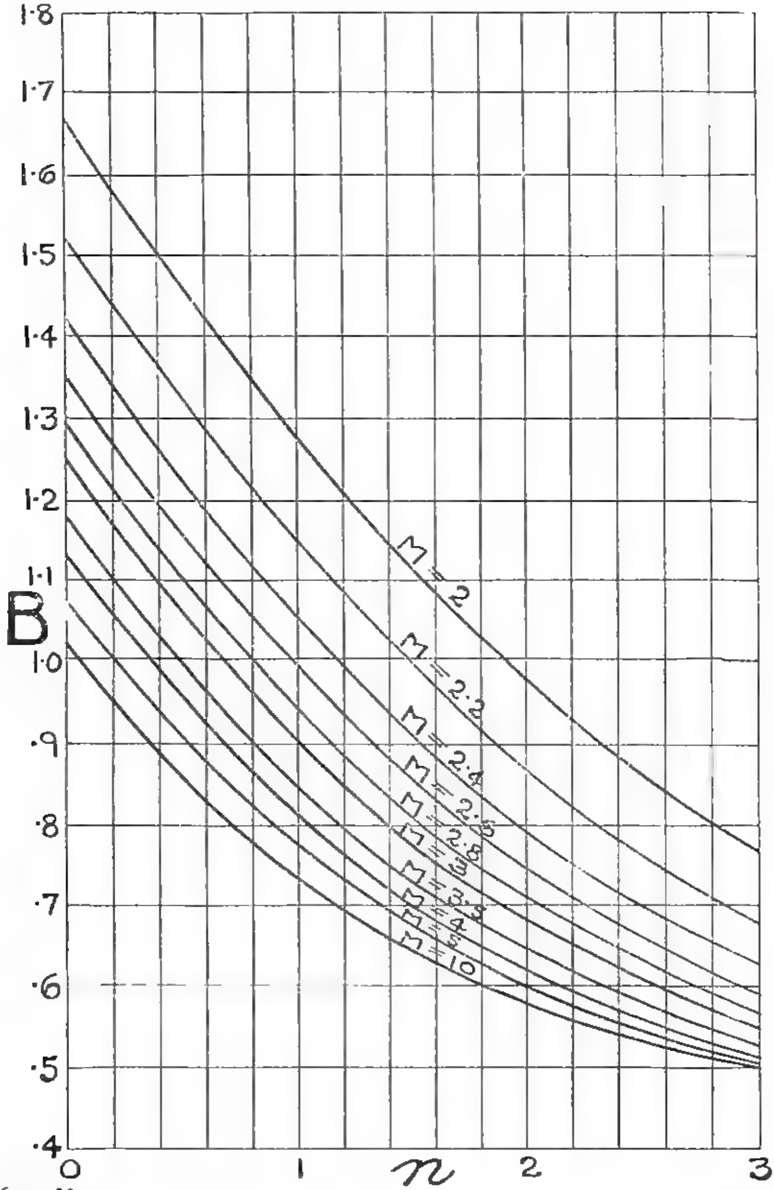
$$\frac{R}{R_0} = M$$

In Eq. (4) of Par. 5, let P be written for p and R for r, and for Y_1 and Y_2 the expressions already found. P will then



"A" CHART FOR DETERMINATION OF P.

FIG. 3.



"B" CHART FOR DETERMINATION OF P.

FIG. 4.

be given by an expression made up of terms consisting of KR^2 multiplied by a constant and terms consisting of Q multiplied by a constant. If, then, the terms be collected into one group of KR^2 terms and another group of Q terms, we may express P in the form—

$$P = AKR^2 + BQ \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

The constant $A =$

$$\frac{[(3+\sigma)\beta_1 - (1+3\sigma)]M^{\alpha_1} - [(3+\sigma)\beta_2 - (1+3\sigma)]M^{\alpha_2} - (3+\sigma) \times (\beta_1 - \beta_2) M^{\alpha_1 + \alpha_2 - 2}}{[8 - (3+\sigma)n]\,[M^{\alpha_1} - M^{\alpha_2}]}$$

and the constant $B =$

$$\frac{\beta_1 M^{\alpha_1} - \beta_2 M^{\alpha_2}}{M^{\alpha_1} - M^{\alpha_2}}$$

Similarly, putting $p = P_0$ and $r = R_0$, we find that P_0 may be expressed :—

$$P_0 = A_0 KR^2 + B_0 Q \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

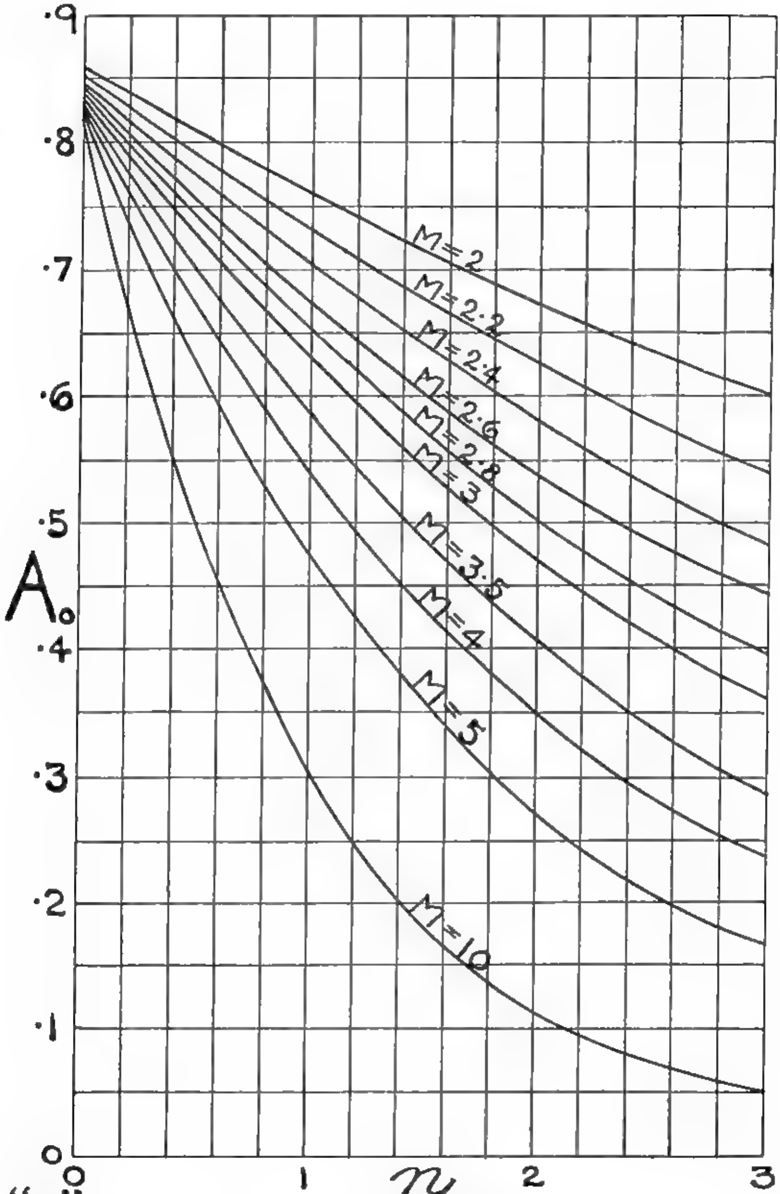
The constant $A_0 =$

$$\frac{(3+\sigma)(\beta_1 - \beta_2)M^2 + [(3+\sigma)\beta_2 - (1+3\sigma)]\,M^{\alpha_1} - [(3+\sigma)\beta_1 - (1+3\sigma)]M^{\alpha_2}}{[8 - (3+\sigma)n]\,M^2\,[M^{\alpha_1} - M^{\alpha_2}]}$$

and the constant $B_0 =$

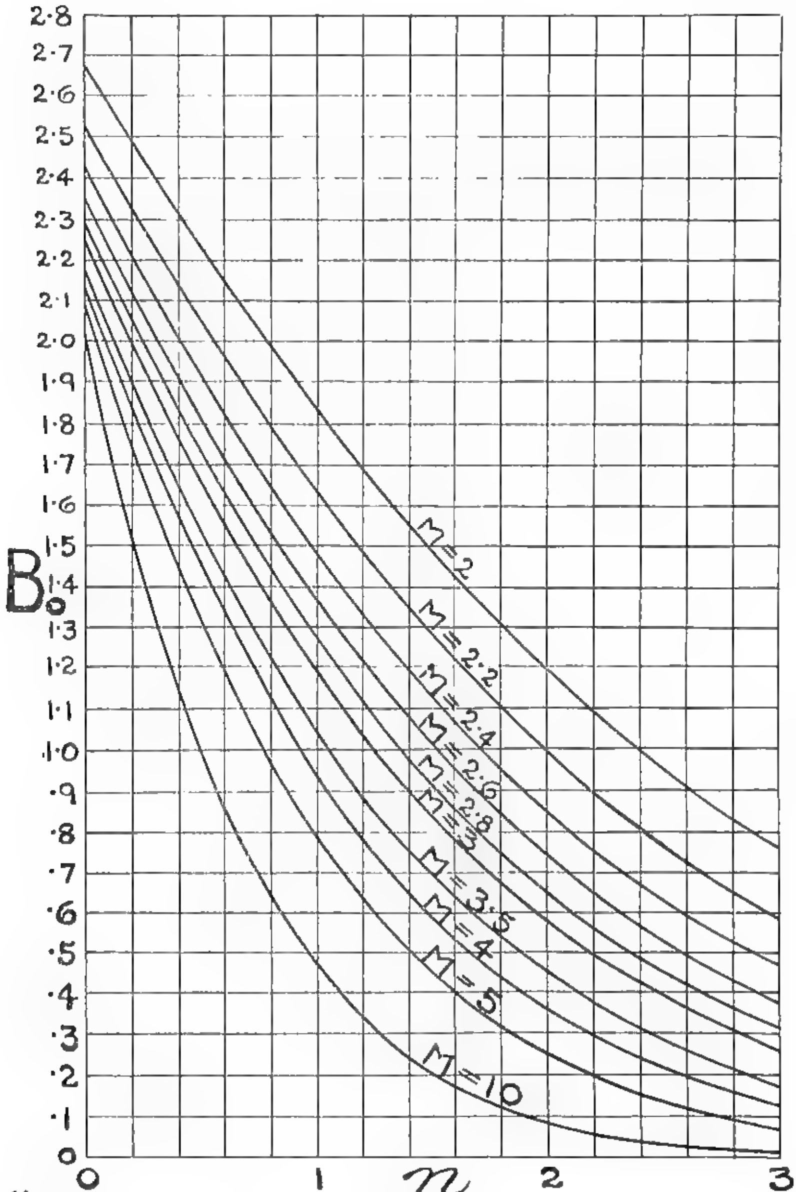
$$\frac{\beta_1 - \beta_2}{M^{\alpha_1} - M^{\alpha_2}}$$

I have worked out the constants A, B, A_0, and B_0 for various values of the ratio M. They are given by the curves in Figs. 3, 4, 5, and 6, for all values of n from 0 to 3. The highest n that is ordinarily required is about 2.5. The higher n is, the greater the variation of disc thickness, and the less is our method of calculation valid, for it is dependent on the assumption that the disc is thin compared with its diameter and that the thickness does not change rapidly anywhere.



"A_o" CHART FOR DETERMINATION OF P_o.

FIG. 5.



"B_o" CHART FOR DETERMINATION OF P_o.

FIG. 6.

8. INDETERMINATE VALUES OF THE CONSTANTS.

In working out the constants A and A_0 a difficulty is encountered when $8-(3+\sigma)\,n$, which occurs in the denominators of A and A_0, becomes zero. It does so when

$$n = \frac{8}{3+\sigma}$$

(For steel, σ may be taken as 0.25, hence the value of n in question is $2\frac{6}{13}$).

Now when
$$n = \frac{8}{3+\sigma},$$

$$\omega = \sqrt{1+\frac{8\sigma}{3+\sigma}+\left(\frac{4}{3+\sigma}\right)^2}$$

$$= \frac{5+3\sigma}{3+\sigma}$$

$$\alpha_1 = -1+\frac{4}{3+\sigma}+\frac{5+3\sigma}{3+\sigma} = 2$$

$$\alpha_2 = -1+\frac{4}{3+\sigma}-\frac{5+3\sigma}{3+\sigma} = -4.\ \frac{1+\sigma}{3+\sigma}$$

$$\beta_1 = -\frac{4}{3+\sigma}+\frac{5+3\sigma}{3+\sigma} = \frac{1+3\sigma}{3+\sigma}$$

$$\beta_2 = -\frac{4}{3+\sigma}-\frac{5+3\sigma}{3+\sigma} = -3$$

By substituting these values of α_1, α_2, β_1, β_2 we find that the numerators of A and A_0 become zero, so that A and A_0 assume the form $\frac{O}{O}$, that is they become vanishing fractions, and are, in an arithmetical sense, indeterminate. We know, however, from the theory of the Differential Calculus, that the fraction $\frac{O}{O}$

may have a perfectly definite value, and it will be seen from Figs. 3 and 5 that there is no discontinuity in the A and A_0 curves where n passes through the value

$$\frac{8}{3+\sigma} = 2\tfrac{6}{13}$$

To find the limiting values of A and A_0 when $n = \dfrac{8}{3+\sigma}$ we proceed as follows :—

Put $$8 - (3+\sigma)n = 2x$$

or $$n = \frac{8}{3+\sigma} - \frac{2x}{3+\sigma}$$

where $2x$ is small, and in the last stage of the process will be put equal to zero.

We have to find $\omega + \delta\omega$, the corresponding value of ω, where $\delta\omega$ is the increment of ω corresponding to the increment

$$\delta n = -\frac{2x}{3+\sigma}$$

$$\delta\omega = \delta\left[1 + \sigma n + \left(\frac{n}{2}\right)^2\right]^{\frac{1}{2}}$$

$$= \frac{1}{2\omega}\left(\sigma + \frac{n}{2}\right)\delta n$$

$$= \frac{1}{2}.\ \frac{3+\sigma}{5+3\sigma}\left(\sigma + \frac{4}{3+\sigma}\right)\left(-\frac{2x}{3+\sigma}\right)$$

$$= -\frac{4+3\sigma+\sigma^2}{(3+\sigma)(5+3\sigma)}.\ x$$

Similarly, we find—

$$\delta\,\beta_1 = \frac{1-\sigma^2}{(3+\sigma)(5+3\sigma)}.\ x$$

$$\delta\,\beta_2 = -\frac{3+\sigma}{5+3\sigma}.\ x$$

Again,

$$\delta M^{\alpha_1} = M^{\alpha_1} (\log_e M)\, \delta\alpha_1$$

$$= M^2 (\log_e M) \left(- \frac{3+\sigma}{5+3\sigma} . x \right)$$

$$= - \frac{3+\sigma}{5+3\sigma} . M^2 (\log_e M)\, x$$

Similarly—

$$\delta M^{\alpha_2} = - \frac{1-\sigma^2}{(3+\sigma)\,(5+3\sigma)} . M^{-4.\frac{1+\sigma}{3+\sigma}} (\log_e M)\, x$$

and in the same way—

$$\delta M^{\alpha_1 + \alpha_2 - 2} = - \frac{2}{3+\sigma} . M^{-4.\frac{1+\sigma}{3+\sigma}} (\log_e M)\, x$$

Now, substituting

$$\beta_1 + \delta\beta_1,\ \beta_2 + \delta\beta_2,\ M^{\alpha_1} + \delta M^{\alpha_1},\ M^{\alpha_2} + \delta M^{\alpha_2},\ M^{\alpha_1 + \alpha_2 - 2}$$
$$+ \delta M^{\alpha_1 + \alpha_2 - 2}$$

for

$$\beta_1,\ \beta_2,\ M^{\alpha_1},\ M^{\alpha_2},\ M^{\alpha_1 + \alpha_2 - 2}$$

in the expressions for A and A_0 (see Par. 7), we find the limiting values of the constants (when $x = 0$) to be:

$$A = \frac{1-\sigma^2}{2(5+3\sigma)} + \frac{(3+\sigma)\, M^{\alpha_2} \log_e M}{M^{\alpha_1} - M^{\alpha_2}}$$

$$A_0 = \frac{1-\sigma^2}{2(5+3\sigma)\, M^2} + \frac{(3+\sigma) \log_e M}{M^{\alpha_1} - M^{\alpha_2}}$$

$$\text{where } \alpha_1 = 2 \text{ and } \alpha_2 = -4.\frac{1+\sigma}{3+\sigma}$$

9. THE STRESS IN A THIN RING.

The formulæ (1) and (2) of Par. 7 give the hoop stresses as the sum of two terms, one proportional to the radial stress Q at the outer radius, the other proportional to KR^2 which is the stress in a radially thin ring of steel of radius R inches rotating at N revs. per min. This may be shewn as follows: Let Fig. 7

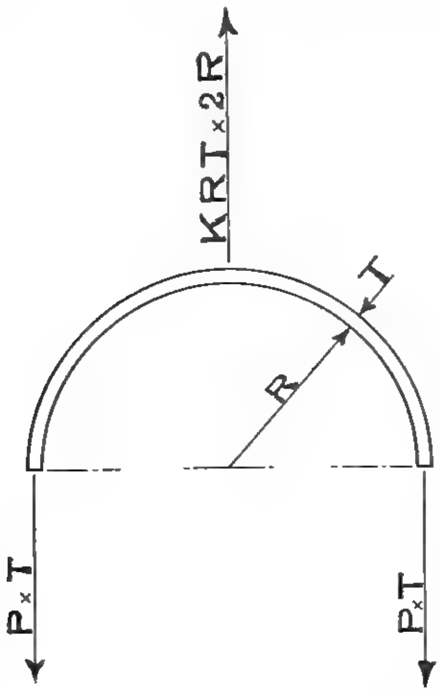


DIAGRAM OF FORCES ON THIN RING.

FIG. 7.

represent a ring of small radial thickness T, 1 inch axial thickness. There is a volume of T cub. inches per inch of circumference, therefore by Par. 3 the centrifugal force per inch of circumference is KRT lbs. This is analogous to boiler pressure, since it acts radially outwards. We thus have—

Pressure × Diameter = 2 × Stress × Thickness, that is—

$$KRT \times 2\ R = 2 \times P \times T$$

whence— $$P = KR^2.$$

10. THIN RING AS LIMITING FORM OF BORED DISC.

If $M=1$, the bored disc has the form of an infinitely thin ring. It will be seen from Par. 7 that in this case B and B_0 become infinite, that is, an infinitely thin ring is incapable of withstanding any superimposed radial stress. And (when $M=1$) A and A_0 again become vanishing fractions.

Put $M=1+x$, x being small, so that its square and higher powers may be neglected, then, by the Binomial Theorem,

$$M^{\alpha_1} = 1 + \alpha_1 x$$

$$M^{\alpha_2} = 1 + \alpha_2 x$$

$$M^2 = 1 + 2x$$

$$M^{\alpha_1+\alpha_2-2} = 1 + (\alpha_1+\alpha_2-2)x$$

The numerator of A becomes—

$$\begin{aligned}&[(3+\sigma)\beta_1-(1+3\sigma)]\alpha_1 x\\ &\qquad -[(3+\sigma)\beta_2-(1+3\sigma)]\alpha_2 x\\ &\qquad\qquad -(3+\sigma)(\beta_1-\beta_2)(\alpha_1+\alpha_2-2)x\end{aligned}$$

and the denominator becomes—

$$[8-(3+\sigma)n](\alpha_1 x-\alpha_2 x)$$

so that the limiting value of

$$A = \frac{-(1+3\sigma)(\alpha_1-\alpha_2) + (3+\sigma)(\alpha_1\beta_2-\alpha_2\beta_1) + 2(3+\sigma)(\beta_1-\beta_2)}{[8-(3+\sigma)n](\alpha_1-\alpha_2)}$$

But

$$\alpha_1-\alpha_2 = \beta_1-\beta_2 = 2\omega$$

and

$$\alpha_1\beta_2-\alpha_2\beta_1 =$$

$$\left(-1+\frac{n}{2}+\omega\right)\left(-\frac{n}{2}-\omega\right) - \left(-1+\frac{n}{2}-\omega\right)\left(-\frac{n}{2}+\omega\right)$$

$$= 2\omega(1-n)$$

Therefore—

$$\Lambda = \frac{-2\omega(1+3\sigma)+2\omega(1-n)(3+\sigma)+4\omega(3+\sigma)}{2\omega\,[8-(3+\sigma)n]}$$

$$= 1.$$

In the same way we find that the limiting value of Λ_0 also $=1$, hence the formulæ (1) and (2) of Par. 7 each reduce to

$$P = KR^2$$

when $M=1$. The stress in this limiting form of the bored disc is thus identical with the stress in a thin ring as found in Par. 9.

11. THE GENERAL SHAPE OF BORED DISCS NOT SUITABLE FOR SOLID DISCS.

It is to be clearly understood that the foregoing investigation relates to discs of ordinary proportions bored through the centre. The type of disc we have considered (defined by Eq. (1) of Par. 4) has no application to solid discs, or even to discs with a very small hole through the centre. It is only necessary to remark that such discs, if they extended to the centre, would there have infinite thickness. There is an important exception, however: When $n=0$. This gives the disc of uniform thickness, a form not often used, but applicable, in favourable conditions, to both solid and bored discs. (See Pars. 15 and 16.)

12. LOAD DUE TO RIM AND BLADING.

The function of a rotor disc is to carry the rim, which, in turn, carries the blades. The ring of blades constitutes a dead load, and the rim, while it takes a certain amount of hoop stress, is, on the whole, a burden on the disc. The actual stress in the rim is not of any particular interest, as it is always lower than the hoop stress in the disc, but it is necessary to take it into account in order to arrive at the stresses in the disc.

Let F = hoop stress in rim.

J = nett section of rim in sq. inches.

T = neck (or least) thickness of disc (inches).

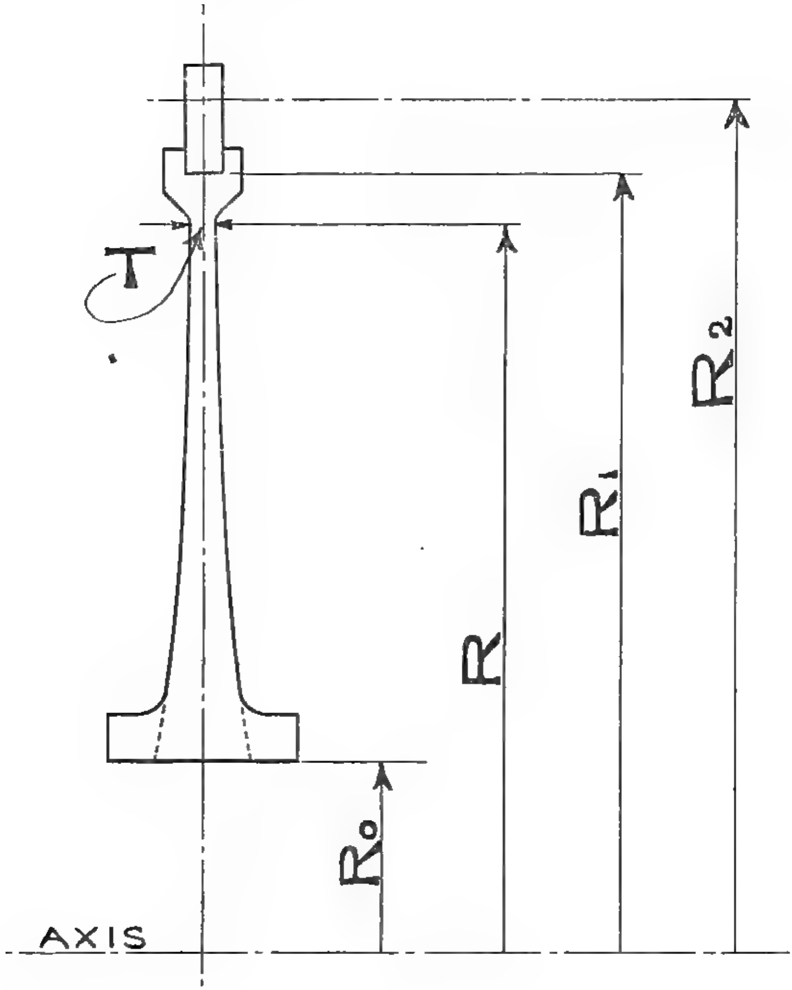
R_1 = mean radius of rim (inches).

R_2 = mean radius of blading (inches).

W_1 = weight of rim (lbs.).

W_2 = weight of blading (lbs.)

(See Fig. 8.)



DIMENSIONS OF BORED DISC.

FIG. 8.

Strictly, R_1 and R_2 should be radii of gyration, not mean radii, but it is convenient and sufficiently exact to take R_1 equal to the radius of the centroid of the rim section, and R_2 equal to the mean radius, commonly called the pitch circle radius, of the blades.

The circumference of the disc being $2\pi R$, the centrifugal force in lbs. per inch of disc circumference due to the rim alone is

$$\frac{W_1}{12 \times 32{\cdot}2} \times R_1 \times \left(\frac{2\pi N}{60}\right)^2 \times \frac{1}{2\pi R}$$

This is equal to

$$\cdot565 \times 8 \times \left(\frac{N}{1000}\right)^2 \times \frac{W_1 R_1}{R}$$

$$= \cdot565\ K.\ \frac{W_1 R_1}{R}$$

Similarly, the centrifugal force in lbs. per inch of disc circumference due to the weight of the blading is

$$\cdot565\ K.\ \frac{W_2 R_2}{R}$$

The total load in lbs. per inch of disc circumference is therefore

$$\cdot565\ K.\ \frac{W_1 R_1 + W_2 R_2}{R}$$

This is an outward force. There is an inward force due to the radial stress Q of amount QT lbs. per inch, so that the nett outward force is

$$\cdot565\ K.\ \frac{W_1 R_1 + W_2 R_2}{R} - QT$$

This acts like a boiler pressure tending to burst the rim, and is resisted by the tensile stress in the rim itself. If we consider a diametral section of the rim (see Par. 9), it is evident that—

$$\left(\cdot565\ K.\ \frac{W_1 R_1 + W_2 R_2}{R} - QT\right) \times 2R = 2 \times F \times J$$

Therefore—

$$QRT + FJ = \cdot565K\ (W_1 R_1 + W_2 R_2) \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

13. HOOP STRAIN IN RIM.

In this problem there are involved four stresses, P, Q, P_0 and F, and so far we have only three equations, namely, (1) and (2) of Par. 7, and (1) of Par. 12. It will be seen that some further condition must be stated before we can solve for any of the stresses.

Now the hoop strain in the disc at its outer radius is, by (1) of Par. 1,

$$\frac{P - \sigma Q}{E}$$

This must be equal to the hoop strain in the rim, that is, to

$$\frac{F}{E}$$

We thus have a fourth equation :

$$F = P - \sigma Q \quad \text{.............................(1)}$$

14. GENERAL SOLUTION FOR BORED DISCS.

The complete statement of the case is, then :—

$P = AKR^2 + BQ$ .... (1), Par. 7.

$P_0 = A_0KR^2 + B_0Q$ .... (2), Par. 7.

$QRT + FJ = .565K \ (W_1R_1 + W_2R_2)$ .... (1), Par. 12.

$F = P - \sigma Q$ .... (1), Par. 13.

To solve these equations for P, Q, P_0 and F, we have from (1) of Par. 7, and (1) of Par. 13—

$$F = AKR^2 + BQ - \sigma Q$$

therefore from (1) of Par. 12—

$$QRT + (AKR^2 + BQ - \sigma Q) \ J = .565K \ (W_1R_1 + W_2R_2)$$

which gives Q. The most convenient form of the solution is—

$$Q = \frac{\cdot 565 . \dfrac{W_1R_1 + W_2R_2}{JR^2} - A}{\dfrac{RT}{J} - \sigma + B} . \ KR^2 \quad \text{..............................(1)}$$

Q being thus found, P is given by (1) of Par. 7, P_0 by (2) of Par. 7, and F by (1) of Par. 13.

Should Q instead of T be assigned at the outset, then P, P_0, and F are at once found, and T is found by transposing (1) of Par. 12, thus:

$$T = \frac{\cdot 565\ K\ (W_1R_1 + W_2R_2) - FJ}{QR} \qquad \ldots\ldots\ \ldots\ldots\ldots\ \ldots\ldots\ldots(2)$$

15. BORED DISC OF UNIFORM THICKNESS.

The bored disc of uniform thickness may be dealt with by the general method, the index n=0, and the case is covered by the charts, Figs. 3, 4, 5, and 6. It may also be treated as a very simple special case.

Referring to Par. 5, when n=0, $\omega = \pm 1$. The effect of taking the negative instead of the positive value would be the same as that of interchanging α_1 and α_2, β_1 and β_2; our final results would be the same whether we take the negative or positive value.

Hence, taking $\omega = +1$, we have $\alpha_1 = 0$, $\alpha_2 = -2$, β_1 -1, $\beta_2 = -1$. Therefore, from Eqs. (3) and (4) of Par. 5—

$$q = Y_1 + \frac{Y_2}{r^2} - \frac{3+\sigma}{8}\ Kr^2 \ \ldots\ldots\ldots\ldots(1)$$

$$p = Y_1 - \frac{Y_2}{r^2} - \frac{1+3\sigma}{8}\ Kr^2 \ \ldots\ldots\ldots(2)$$

From (1):—

$$O = Y_1 + \frac{Y_2}{R_0^2} - \frac{3+\sigma}{8}\ KR_0^2 \ \ldots\ldots\ldots\ldots(3)$$

$$Q = Y_1 + \frac{Y_2}{R^2} - \frac{3+\sigma}{8}\ KR^2 \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Subtracting (4) from (3):

$$-Q = Y_2\left(\frac{1}{R_0^2} - \frac{1}{R^2}\right) + \frac{3+\sigma}{8}\ (KR^2 - KR_0^2)$$

$$= \frac{Y_2}{R^2}\ (M^2 - 1) + \frac{3+\sigma}{8}\left(1 - \frac{1}{M^2}\right)\ KR^2$$

whence—

$$\frac{Y_2}{R^2} = -\frac{3+\sigma}{8}.\frac{KR^2}{M^2} - \frac{Q}{M^2-1}$$

and from (3)—

$$Y_1 = \frac{3+\sigma}{8}.\ KR_0^2 - \frac{Y_2}{R_0^2}$$

$$= \frac{3+\sigma}{8}\left(1+\frac{1}{M^2}\right)KR^2 + \frac{M^2}{M^2-1}.\ Q$$

Substituting in (1) and (2):

$$q = \frac{3+\sigma}{8}\left(1+\frac{1}{M^2} - \frac{1}{M^2}.\frac{R^2}{r^2} - \frac{r^2}{R^2}\right)KR^2 + \left(M^2 - \frac{R^2}{r^2}\right)\frac{Q}{M^2-1}$$

$$p = \frac{3+\sigma}{8}\left(1+\frac{1}{M^2} + \frac{1}{M^2}.\frac{R^2}{r^2} - \frac{1+3\sigma}{3+\sigma}.\frac{r^2}{R^2}\right)KR^2 + \left(M^2 + \frac{R^2}{r^2}\right)\frac{Q}{M^2-1}$$

$p=P$ where $r=R$, and $=P_0$ where $r=R_0$:

$$P = \frac{3+\sigma}{4}\left(\frac{1-\sigma}{3+\sigma} + \frac{1}{M^2}\right)KR^2 + \frac{M^2+1}{M^2-1}.\ Q \qquad \text{(5)}$$

$$P_0 = \frac{3+\sigma}{4}\left(1+\frac{1}{M^2}.\frac{1-\sigma}{3+\sigma}\right)KR^2 + \frac{2M^2}{M^2-1}.\ Q \qquad \text{(6)}$$

If the hole through the centre is infinitely small then $M =$ infinity, and

$$P = \frac{1-\sigma}{4}.\ KR^2 + Q \qquad \text{(7)}$$

$$P_0 = \frac{3+\sigma}{4}.\ KR^2 + 2\,Q \qquad \text{(8)}$$

This not the case of a solid disc, however, because it is based on the assumption that q—O at the bore, nor is it of use for design purposes; it is given for comparison with the case of a solid flat disc. (See Par. 16.)

16. SOLID DISC OF UNIFORM THICKNESS.

In a solid disc, the radial and hoop stresses at the centre are indistinguishable, that is, $P_0 = Q_0$. Considering a point very near the centre of the disc, then, equating (1) and (2) of Par. 15,—

$$\frac{Y_2}{r^2} - \frac{3+\sigma}{8}.\ K r^2 = -\frac{Y_2}{r^2} - \frac{1+3\sigma}{8}.\ Kr^2$$

Multiplying throughout by r^2 and then putting $r = O$, we deduce

$$Y_2 = O$$

and therefore from (4) of Par. 15,—

$$Y_1 = \frac{3+\sigma}{8}.\ KR^2 + Q$$

hence, substituting in (1) and (2) of Par. 15,—

$$q = \frac{3+\sigma}{8}(KR^2 - Kr^2) + Q$$

$$p = \frac{3+\sigma}{8}.\ KR^2 - \frac{1+3\sigma}{8}.\ Kr^2 + Q$$

At the outer radius:

$$P = \frac{1-\sigma}{4}.\ KR^2 + Q \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and at the centre:

$$P_0 = \frac{3+\sigma}{8}.\ KR^2 + Q \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Now comparing (1) and (2) with (7) and (8) of Par. 15, we see that the hoop stresses at the outer radii of a flat disc with an infinitely small bore and a flat solid disc are equal (with, of course,

a given KR^2 and Q). But the hoop stress at the centre of the disc with infinitely small bore is exactly double the stress at the centre of the solid disc.

17. DISC OF UNIFORM STRENGTH.

It was noted in Par. 3 that a solid disc may be designed to have the radial and hoop stresses equal and uniform, such condition being compatible with the strain equation (3) of Par. 1. Writing the equation of equilibrium (1) of Par. 2 in accordance with the condition—

$$p = q = \text{constant} = Q$$

we have—

$$\begin{aligned}(Q - Kr^2)\ t\ dr &= Qd\,(rt)\\ &= Q\,(rdt + t\,dr)\end{aligned}$$

whence $Qrdt = -Kr^2t\,dr$

or

$$\frac{dt}{t} = -\frac{K}{Q}.\,rdr$$

Integrating,

$$\log_e t = -\frac{Kr^2}{2\,Q} + \text{constant}$$

If T_0 is the thickness at the centre (where $r = O$) then

$$\text{the constant} - \log_e T_0$$

$$\log_e \frac{t}{T_0} = -\frac{Kr^2}{2\,Q}$$

Let

$$\gamma = \frac{K}{2\,Q}$$

Then

$$\log_e \frac{t}{T_0} = -\gamma r^2$$

or

$$t = T_0 e^{-\gamma r^2}$$

The thickness T at the outer radius R is

$$T = T_0 e^{-\gamma R^2}$$

whence

$$T_0 = T e^{\gamma R^2}$$

and therefore—

$$t = T e^{\gamma (R^2 - r^2)} \quad \text{.......................(1)}$$

Eq. (1) of Par. 13 becomes—

$$F = Q - \sigma Q$$

or putting $\sigma = .25$, we have stress in rim

$$F = .75\,Q$$

Eq. (1) of Par 12 becomes—

$$QRT + .75\,QJ = .565K\,(W_1\ R_1 + W_2\ R_2)$$

so that, if T is assumed,—

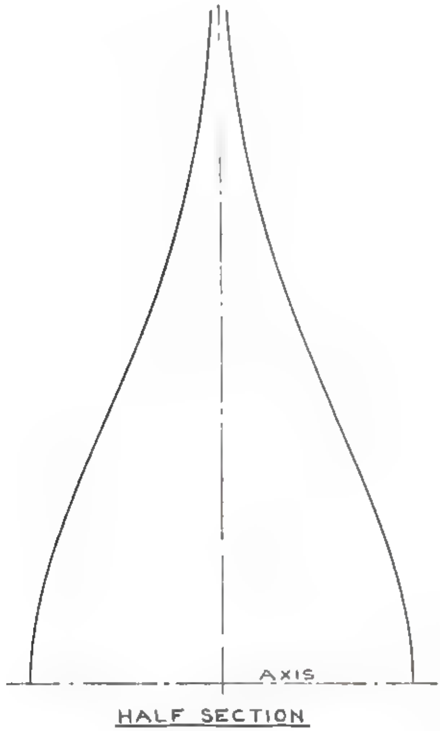
$$Q = .565\ K. \frac{W_1R_1 + W_2R_2}{RT + .75\ J} \quad \text{..................(2)}$$

Or, if Q is assigned instead of T,—

$$T - .565\,\frac{K}{Q}.\ \frac{W_1R_1 + W_2R_2}{R} - .75\,\frac{J}{R} \quad \text{...........(3)}$$

The disc of uniform strength has the form shown (exaggerated) in Fig. 9. It will be seen that the direction of curvature of the profile is reversed. The point of reversal is given by the condition—

$$\frac{d^2t}{dr^2} = O \quad \text{.......................(4)}$$



DISC OF UNIFORM STRENGTH.

FIG. 9.

Since

$$\frac{dt}{dr} = - 2\gamma T_0 re^{-\gamma r^2}$$

we find—

$$\frac{d^2t}{dr^2} = 2\gamma T_0(2\gamma r^2 - 1)\, e^{-\gamma r^2}$$

whence, by (4), the radius of the point of reversal is

$$r = \sqrt{\frac{1}{2\gamma}} = \sqrt{\frac{Q}{K}}$$

18. ROTOR DRUM.

This type of rotor may be used in reaction turbines if the blade speed is relatively low. Its treatment does not strictly come within our subject, but the hoop stress in the drum is readily deduced from Eq. (1) of Par. 12, the drum being treated as a thin ring.

Let—

W_1 = weight of drum in lbs. per inch length of drum.
W_2 = weight of blading in lbs. per inch length of drum.
R_1 = mean radius of drum in inches.
R_2 = mean radius of blading in inches.
J = mean thickness of drum in inches.
F = hoop stress in drum.

Then it may be shown that

$$F = .565K. \frac{W_1R_1 + W_2R_2}{J}$$

Compare this with Eq. 1 of Par. 12.

19. MINIMUM THICKNESS OF DISC.

The neck or least thickness of an impulse turbine disc should not be less than a certain minimum however low the stress is. A good rule is to make this minimum equal to

$$\frac{R_2}{60}$$

but not less than $\frac{1}{4}''$ however small is R_2, otherwise there is a chance of accidental distortion when the disc is handled in the shops.

20. SUMMARY OF FORMULÆ FOR DISCS.

I. *Discs with Central Bore.*

$T = \dfrac{R_2}{60}$ or $T = \frac{1}{4}''$ whichever is the greater. (Par. 19.)

Take a trial value of n (this will be known approximately, from experience, for different types of rotor), and from the charts Figs. 3 and 4 obtain A and B.

$$Q = \frac{.565\,\dfrac{W_1R_1+W_2R_2}{JR^2} - A}{\dfrac{RT}{J} - .25 + B} \cdot KR^2$$

.... (1), Par. 14.

$$P = AKR^2 + BQ \quad \text{.......................(1), Par. 7.}$$

If the stresses P and Q are satisfactory, proceed to obtain A_0 and B_0 from Figs. 5 and 6.

$$P_0 = A_0KR^2 + B_0\,Q \quad \text{.......................(2), Par. 7.}$$

Also, $$F = P - .25Q \quad \text{.......................(1), Par. 13.}$$

but it may not be necessary to take out F, which is always less than P.

If Q, thus found, is too high, then T should be increased. If P or P_0 is too high, then a higher value of n must be tried. If P and P_0 are both lower than they need be, a lower n may be tried. Should Q instead of T be determined on at the outset, find n, A. B, A_0, B_0, and calculate P, P_0, and F. Then—

$$T = \frac{.565\,K\,(W_1R_1+W_2R_2) - FJ}{QR}$$

.......................(2), Par. 14.

The thickness of this type of disc at any radius r is

$$t = \frac{C}{r^n} \quad \text{.......................(1), Par. 4.}$$

and the constant $C = TR^n$

II. *Flat Disc with Central Bore.*

This must have a thickness not less than, say, $\frac{3}{4}''$.

In this case, n = O. A and B are found from Figs. 3 and 4, or calculated from the simple formulæ—

$$\left.\begin{aligned} A &= \frac{13}{16}\left(\frac{3}{13} + \frac{1}{M^2}\right) \\ B &= \frac{M^2+1}{M^2-1} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(5), \text{ Par. } 15.$$

Q is then found as in I.

If Q is too high, T must be increased.

P is found as in I.

A_0 and B_0 may be read from Figs. 5 and 6, or obtained from the formulæ—

$$\left.\begin{aligned} A_0 &= \frac{13}{16}\left(1 + \frac{3}{13\, M^2}\right) \\ B_0 &= \frac{2\, M^2}{M^2 - 1} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(6), \text{ Par. } 15.$$

P_0 is then found as in I.

If P or P_0 is too high, T must be increased.

If Q is assumed instead of T, find A, B, A_0, B_0, and proceed as in I.

III. *Flat Solid Disc.*

Figs. 3, 4, 5 and 6 do not apply.

$$\left.\begin{aligned} A &= \frac{3}{16} \\ B &= 1 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(1), \text{ Par. } 16.$$

$$\left.\begin{aligned} A_0 &= \frac{13}{32} \\ B_0 &= 1 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(2), \text{ Par. } 16.$$

Otherwise the procedure is the same as in II.

IV. *Disc of Uniform Strength.*

The least thickness T may be fixed as in I.

$$Q = \cdot 565\, K . \frac{W_1 R_1 + W_2 R_2}{RT + \cdot 75\, J}$$

........................(2), Par. 17.

If Q is predetermined,—

$$T = \cdot 565 \frac{K}{Q} . \frac{W_1 R_1 + W_2 R_2}{R} - \frac{\cdot 75\, J}{R}$$

........................(3), Par. 17.

Also, $F = .75\, Q$.

The thickness of the disc at any radius r is

$$t = Te^{\gamma(R^2 - r^2)}$$ .... (1) Par. 17.

where

$$\gamma = \frac{K}{2Q}$$

21. EXAMPLE 1.

A disc carrying blades 45″ mean diameter is to run at 3300 revs. per min. The bore diameter is 10″, and the neck radius is 19″. Mean radius of rim is 20″, nett section area 2.6 sq. ins. The weight of the rim is 80 lbs., and of the blading 70 lbs. The stress is not to exceed 16,000 lbs. per sq. in. Find the shape of the disc.

$$M = \frac{R}{R_0} = \frac{19}{5} = 3 \cdot 8$$

$$K = 8 \times \left(\frac{N}{1000}\right)^2 = 8 \times 3.3^2 = 87$$

$$KR^2 = 87 \times 19^2 = 31400$$

$$W_1\ R_1 = 80 \times 20 \quad = 1600$$
$$W_2\ R_2 = 70 \times 22.5 = 1575$$

$$W_1\ R_1 + W_2\ R_2 \quad = 3175$$
$$JR^2 = 2.6 \times 19^2 = 940$$

$$\cdot 565 \times \frac{W_1R_1+W_2R_2}{JR^2} = .565 \times \frac{3175}{940} = 1.91$$

$$T = \frac{R_2}{60} = \frac{22.5}{60} = .375$$

$$\frac{RT}{J} = \frac{19 \times \cdot 375}{2.6} = 2\cdot 74$$

Subtract : .25

$$\frac{RT}{J} - .25 = 2.49$$

$$Q = \frac{\cdot 565\,\frac{W_1R_1+W_2R_2}{JR^2} - A}{\frac{RT}{J} - .25 + B} \times KR^2$$

$$= \frac{1\cdot 91 - A}{2\cdot 49 + B} \times 31400$$

Taking a trial value of n=2, we find from Figs. 3 and 4—

$$A=.14,\ B=.63$$

$$Q = \frac{1\cdot 77}{3.12} \times 31400 = 17800 \text{ lbs. per sq. in.}$$

which is beyond the limit of 16000 allowed in this case

It is therefore necessary to make T greater than the minimum of $\frac{3}{8}''$: let $T=\frac{7}{16}''=.4375$.

Then

$$\frac{RT}{J} = \frac{19 \times \cdot 4375}{2.6} = 3.20$$

Subtract .25

$$\frac{RT}{J} - .25 = 2.95$$

With n (as before) = 2,—

$$Q = \frac{1.91. - .14}{2.95 + .63} \times 31400 = 15500 \text{ lbs. per sq. in.}$$

$$AKR^2 = .14 \times 31400 = 4400$$
$$BQ = .63 \times 15500 = 9800$$

$$P = AKR^2 + BQ = 14200 \text{ lbs. per sq. in.}$$

From Figs. 5 and 6, $A_0 = .37$, $B_0 = .39$:

$$A_0KR^2 = .37 \times 31400 = 11600$$
$$B_0Q = .39 \times 15500 = 6050$$

$$P_0 = A_0KR^2 + B_0Q - 17650 \text{ lbs. per sq. in.}$$

This stress, again, is too high, and we have to try a higher index in addition to the increased neck thickness of $\frac{7}{16}''$.

Assume n = 2.2, then from the charts,

$$A = .13,\ B = .60,\ A_0 = .34, \text{ and } B_0 = .33.$$

$$Q = \frac{1.91 - .13}{2.95 + .60} \times 31400 = 15700 \text{ lbs. per sq. in.}$$

$$AKR^2 = .13 \times 31400 = 4100$$
$$BQ = .60 \times 15700 = 9400$$

$$P = 13500 \text{ lbs. per sq. in.}$$

$$A_0KR^2 = .34 \times 31400 = 10700$$
$$B_0Q = .33 \times 15700 = 5200$$

$$P_0 = 15900 \text{ lbs. per sq. in.}$$

Hence, a neck thickness of $\frac{7}{16}''$ and an index 2.2 will be satisfactory. To get the shape of the disc—

$$C = TR^n = .4375 \times 19^{2 \cdot 2}$$
$$= 281.6$$

and the thickness at any radius r is

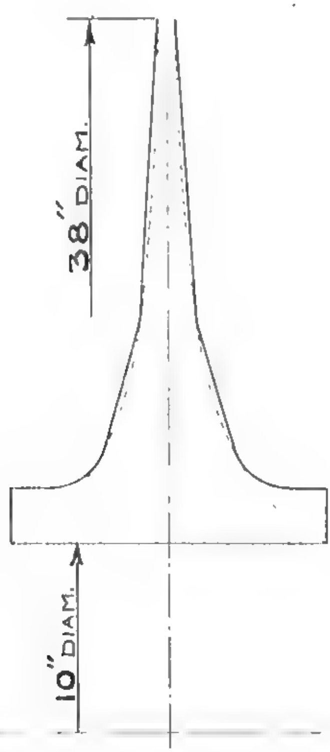
$$t = \frac{C}{r^n} = \frac{284 \cdot 6}{r^{2 \cdot 2}}$$

From this the following figures are obtained :—

r ...	19	15	11	9	8
t ...	.1375	.736	1.156	2.264	2.934

r ...	7	6.5	6	5.5	5
t ...	3.936	4.633	5.525	6.690	8.251

The disc is drawn to scale in Fig. 10. The calculated disc is shown by the dotted lines, the full lines representing the form to which the disc may be machined.



BORED DISC.—EXAMPLE 1.

FIG. 10.

22. EXAMPLE 2.

The mean diameter of the blading of a lightly loaded disc is 28″. Weight of blading 25 lbs. Rim weight 19 lbs., mean radius 12.5″, section area .85 sq. ins. Neck radius 11.5″, bore 8.25″ diameter. Revolutions 2000 per min. Find the stresses, (a) if the disc is made according to an index of .6, with a neck thickness of .25″, (b) if the disc has a uniform thickness of .75″.

$$M = \frac{11\cdot5}{4\cdot125} = 2.8$$

$$K = 8 \times 2^2 = 32$$
$$KR^2 = 32 \times 11.5^2 = 4232$$

$$W_1 R_1 = 19 \times 12.5 = 238$$
$$W_2 R_2 = 25 \times 14 \quad = 350$$

$$W_1 R_1 + W_2 R_2 = 588$$

$$\cdot565 \times \frac{W_1R_1 + W_2R_2}{JR^2} = \frac{\cdot565 \times 588}{\cdot85 \times 11\cdot5^2} = 2\cdot96$$

(a) From the charts, $A = .26$, $B = 1.07$, $A_0 = .72$, $B_0 = 1.63$.

$$\frac{RT}{J} = \frac{11\cdot5 \times \cdot25}{\cdot85} = 3\cdot39$$

Subtract: .25

3·14

$$Q = \frac{2\cdot96 - \cdot26}{3\cdot14 + 1\cdot07} \times 4232 = 2710 \text{ lbs. per sq. in.}$$

$$AKR^2 = .26 \times 4232 = 1100$$
$$BQ = 1.07 \times 2710 = 2900$$

$P = 4000$ lbs. per sq. in.

$$A_0KR^2 = .72 \times 4232 = 3050$$
$$B_0Q = 1.63 \times 2710 = 4420$$

$P_0 = 7470$ lbs. per sq. in.

(b) From the charts (with n=O) or by calculation, $A=.29$, $B=1.29$, $A_0=.81$, $B_0=2.29$.

$$\frac{RT}{J} = \frac{11.5 \times .75}{.85} = 10.15$$

$$\begin{array}{rr} \text{Subtract :} & .25 \\ \hline & 9.90 \end{array}$$

$$Q = \frac{2.96 - .29}{9.90 + 1.29} \times 4232 = 1010 \text{ lbs. per sq. inch.}$$

$$\begin{array}{r} AKR^2 = \ .29 \times 4232 = 1230 \\ BQ = 1.29 \times 1010 = 1300 \\ \hline P = 2530 \text{ lbs. per sq. in.} \end{array}$$

$$\begin{array}{r} A_0KR^2 = \ .81 \times 4232 = 3550 \\ B_0Q = 2.29 \times 1010 = 2310 \\ \hline P_0 = 5860 \text{ lbs. per sq. in.} \end{array}$$

23. EXAMPLE 3.

A solid disc is $\frac{7}{8}''$ uniform thickness. The blading is 34″ mean diameter, 50 lbs. weight. The rim is 15″ mean radius, 2.4 sq. ins. section, and 54 lbs. weight. Neck radius 14″. Revolutions 3500. Find the stresses at the neck, and the stress at the centre.

We have $A = \frac{3}{16}$, $B = 1$, $A_0 = 13/32$, $B_0 = 1$.

$$K = 8 \times 3.5^2 = 98$$

$$KR^2 = 98 \times 14^2 = 19200$$

$$\begin{array}{r} W_1 R_1 = 54 \times 15 = 810 \\ W_2 R_2 = 50 \times 17 = 850 \\ \hline W_1 R_1 + W_2 R_2 = 1660 \end{array}$$

$$JR^2 = 2.4 \times 14^2 = 470$$

$$.565 \times \frac{1660}{470} = 2.00$$

$$\begin{array}{rr} \text{Subtract } A = \frac{3}{16} = & .19 \\ \hline & 1.81 \end{array}$$

$$\frac{RT}{J} = \frac{14 \times .875}{2.4} = 5.10$$

Subtract: .25

4.85

Add B = 1.00

5.85

$$Q = \frac{1.81}{5.85} \times 19200 = 5950 \text{ lbs. per sq. in.}$$

$$AKR^2 = \tfrac{3}{16} \times 19200 = 3600$$
$$BQ = 1 \times 5950 = 5950$$

P = 9550 lbs. per sq. in.

$$A_0KR^2 = \tfrac{13}{32} \times 19200 = 7800$$
$$B_0Q = 1 \times 5950 = 5950$$

$P_0 = 13750$ lbs. per sq. in.

24. EXAMPLE 4.

The spindle of a solid rotor is 10″ diameter. The mean diameter of blading is 36″, the revolutions 4500. The blading weight is 45 lbs. The rim, 33″ mean diameter, 3.2 sq. ins. section, is 70 lbs. weight. The neck is 15.5″ radius, ½″ thickness. Find the shape of the disc, which is to be uniformly stressed.

$$K = 8 \times 1.5^2 = 162$$

$$W_1 R_1 = 70 \times 16.5 = 1155$$
$$W_2 R_2 = 45 \times 18 = 810$$

1965

$$RT = 15.5 \times .5 = 7.75$$
$$.75J = 75 \times 3.2 = 2.40$$

10.15

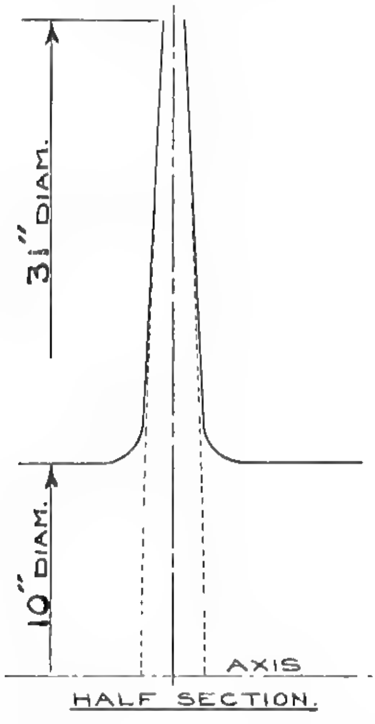
$$Q = \frac{.565 \times 162 \times 1965}{10.15}$$

= 17700 lbs. per sq. in.

$$\gamma = \frac{K}{2Q} = \frac{162}{2 \times 17700} = \frac{1}{219}$$

$$t = Te^{\gamma(R^2 - r^2)}$$

$$\text{or} \quad \log_{10} t = \log_{10} .5 + \frac{15.5^2 - r^2}{219} \times \log_{10} e$$



SOLID DISC.—EXAMPLE 4.

FIG. 11.

Hence the following thicknesses may be calculated :—

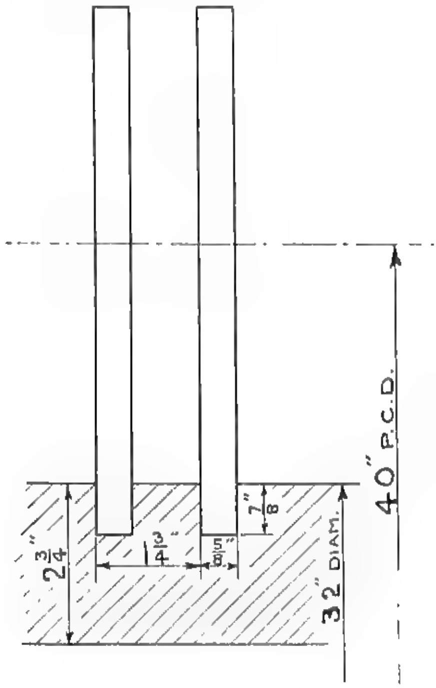
r	15.5	12.5	10.5	9	8
t	.5	.731	.899	1.028	1.109

7	6	5	4	2	0
1.188	1.259	1.323	1.380	1.455	1.481

The dotted lines in Fig. 11 show the calculated outline of the disc, and the full lines show the simplified form to which the disc is turned.

25. EXAMPLE 5.

The longest blades of a reaction turbine rotor are 10″ mean diameter. The rows are $1\frac{3}{4}''$ pitch, each row having a weight of



SECTION OF DRUM AND BLADING OF REACTION TURBINE — EXAMPLE 5

FIG. 12.

110 lbs. The drum is 32″ outside diameter and $2\frac{3}{4}''$ thick, and the grooves are $\frac{5}{8}''$ wide × $\frac{7}{8}''$ deep. What is the stress in the drum when the rotor runs at 1500 revs.? (See Fig. 12.)

$$\begin{array}{rl} 1\frac{3}{4} \times 2\frac{3}{4} = 1.81 & \text{sq. ins.} \\ \frac{3}{4} \times \frac{3}{4} = .55 & \text{,,} \quad \text{,,} \\ \hline 4.26 & \end{array}$$

Therefore, mean thickness of drum =

$$\frac{4.26}{1.75} = 2.44''$$

Mean radius of drum approximately =

$$\frac{32 - 2.44}{2} = 14.8''$$

Weight of drum, lbs. per inch length of drum =

$$2\pi \times 14.8 \times 2.44 \times .28 = 63.5$$

Weight of blading in lbs. per inch length of drum =

$$\frac{140}{1.75} = 80$$

$$K = 8 \times 1.5^2 = 18$$

$$\begin{array}{l} W_1 R_1 = 63.5 \times 14.8 = 910 \\ W_2 R_2 = 80 \times 20 = 1600 \\ \hline 2510 \end{array}$$

$$F = .565 \times 18 \times \frac{2510}{2.44}$$

$$= 10600 \text{ lbs. per sq. in.}$$

26. STRESS IN RIM FLANKS.

A severe stress often arises in the rim flanks of impulse turbine rotors. Referring to Fig. 13, and considering 1 inch length of the rim, then, if the centrifugal load λ lbs. per inch is taken equally by the two flanks, there is a bending moment $\frac{1}{2}\lambda l$ on each flank in addition to the direct tension $\frac{1}{2}\lambda$. The latter produces the direct stress—

$$\frac{\frac{1}{2}\lambda}{V}$$

while the bending stress is—

$$\frac{\frac{1}{2}\lambda U}{\frac{1}{6}V^2}$$

Hence the combined stress in the flanks is

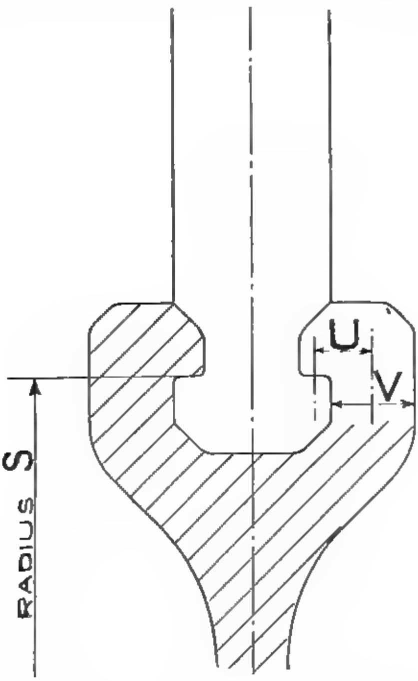
$$\frac{\frac{1}{2}\lambda}{V} + \frac{\frac{1}{2}\lambda U}{\frac{1}{6}V^2}$$

$$= \frac{\lambda}{2V}\left(1+6\,\frac{U}{V}\right)$$

From Par. 12 it will be seen that the load per inch

$$\lambda = .565K\,.\,\frac{W_2\,R_2}{S}$$

S being the groove radius as indicated in Fig. 13.



DIMENSIONS FOR CALCULATING

FLANK STRESS.

FIG. 13.

EXAMPLE 6.

A row of blades 50″ mean diameter has a weight of 198 lbs. The blades are $1\frac{1}{8}$″ wide, the neck of the root is $\frac{7}{8}$″, and the groove radius S is 20″. The width of the rim is $2\frac{5}{8}$″. The revolutions are 2500. Examine the strength of the flanks.

$$K = 8 \times 2.5^2 = 50$$

$$\lambda = .565 \times 50 \times \frac{198 \times 25}{20}$$

$$= 7000 \text{ lbs. per inch.}$$

In Fig. 13 the flank V is equal to

$$\tfrac{1}{2}\,(\text{Rim width—Blade width});$$

therefore $V = \dfrac{2\frac{5}{8} - 1\frac{1}{8}}{2} = .75.$ The arm U is $\frac{1}{2}$ (Shoulder + Flank)

and (Shoulder + Flank) is equal to $\frac{1}{2}$ (Rim width − Neck), therefore

$$U = \tfrac{1}{4}\,(\text{Rim width} - \text{Neck})$$

$$= \frac{2\frac{5}{8} - \frac{7}{8}}{4} = .4375$$

$$6 \times \frac{U}{V} = 6 \times \frac{.4375}{.75} = 3.5$$

Therefore, stress in flanks =

$$\frac{7000}{2 \times .75} \times 4.5 = 21000 \text{ lbs per sq. in.}$$

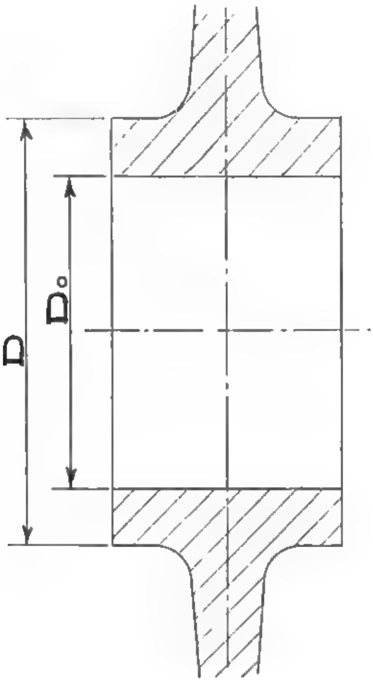
This stress is higher than is usually allowed in other parts of a disc, and it would be desirable to reduce it by increasing the rim width.

Some designers allow very high flank stresses—I have noted stresses, calculated on this basis, approaching 30000 lbs. per sq. inch. Considering that turbines may be run above their designed speeds, and that all centrifugal stresses increase as the square of the speed, it is most advisable that stresses be kept below so high a value which gives so little margin under the elastic limit. Where possible, flank stress should be limited to 12000 lbs. per sq. in.

27. FORCE FIT OF WHEEL ON SPINDLE.

Due to strain when a disc rotates, the bore of the hub enlarges much more than does the shaft on which the disc is mounted, and, unless the disc is given initially a tight fit, it may become loose when rotating.

The centrifugal strain at the bore is $\frac{P_0}{E}$, therefore if the disc is



DIMENSIONS FOR CALCULATING
SHRINKAGE STRESS.

FIG. 14.

mounted on the shaft with a force fit or shrinkage strain δ rather greater than $\frac{P_0}{E}$ it will remain tight during rotation. (δ, of course, is a ratio; it is sometimes expressed as so much of an "inch, per inch.") When the rotor is stopped, the strain δ is divided

between the hub and the shaft; the hub is in tension, the shaft in compression. The tensile stress in the hub is greater than the compressive stress in the shaft. Again, the stress in the cylindrical ends of the boss (the weakest parts of the hub) is greater than the stress near the middle of the boss; but even at the ends of the boss the stress is less than $E \times \delta$, on account of the compression of the shaft.

(See Fig. 14.)

Let D_0 = diameter of shaft
D = ,, boss

and
$$\mu = \frac{D_0}{D}$$

then it may be shown that, when the rotor is at rest, the stress in the cylindrical ends of the boss is

$$E \times \delta \times \frac{1+\mu^2}{2} \text{ lbs. per sq. in.}$$

EXAMPLE 7.

The centrifugal hoop stress at the bore of a rotor disc is 18000 lbs. per sq. in. The disc has a bore of $15\frac{1}{4}''$, and the boss diameter is $18''$. The seat has a taper of $\frac{1}{4}''$ per foot on diameter. Determine a suitable driving allowance, and the initial stress in the boss.

The shrinkage strain should be greater than

$$\frac{P_0}{E} = \frac{18{,}000}{30{,}000{,}000} = .0006$$

that is, $.0006''$ per inch of diameter. Therefore the corresponding allowance on the diameter is

$$.0006 \times 15.25 = .00915''$$

$\frac{1}{4}''$ per ft. is 1 in 48, hence the corresponding driving allowance is

$$.00915 \times 48 = .44''$$

Make it $\frac{1}{2}''$, then the allowance on diameter is

$$\frac{.5}{48} = .0104''$$

and the strain

$$\delta = \frac{.01042}{15.25} = .00068$$

$$\mu = \frac{15.25}{18} = \cdot 85$$

$$\mu^2 = .72$$

$$\frac{1+\mu^2}{2} = \frac{1.72}{2} = .86$$

The initial stress in the boss is therefore

$$30,000,000 \times .00068 \times .86$$
$$= 17500 \text{ lbs. per sq. in.}$$

28. PRACTICAL CONSIDERATIONS.

All our rules for discs of varying thickness are dependent on the assumption that the thickness varies slowly. For that reason, in discs of the form tr^n =constant, the index n should not, if possible, be higher than 2. In discs of uniform strength, the thickness at the centre should not be too disproportionate with the thickness at the neck. In De Laval discs where the variation of thickness is great, there is extra thinning below the rim so that fracture if any will occur there and allow the rim with the blading to come to rest, the centre part of the disc being thus relieved.

Regarding the magnitude of stress that may be allowed in rotor discs, this may be as high as 25000 lbs. per sq. inch, with suitably good material. With 34 to 38 ton steel, as is used in marine work, a maximum stress of 18000 lbs. per sq. in. is certainly low enough. Sharp corners or edges on a rotor disc should be avoided. The rim should be well rounded, and the disc proper well radiused to the rim and to the boss. There should be no tool marks on the finished disc—a scratch may be the genesis of a fracture—the finish should be dead smooth. It is the excellent practice of at least one firm building turbo-generators to leave the discs highly polished.

29. APPENDIX.

(a) The volume of a disc is very simply found when

$$tr^n = TR^n = \text{constant} = C$$

Volume

$$= \int_{R_0}^{R} 2\pi r\, t\, dr$$

$$= 2\pi C \int_{R_0}^{R} r^{1-n}\,dr$$

$$= 2\pi C \; \frac{R^{2-n} - R_0^{2-n}}{2-n}$$

If $n=0$, then $C=T$, and

$$\text{Volume} = \pi T\,(R^2 - R_0^2)$$

If $n = 1$, $C = R.T$:

$$\text{Volume} = 2\pi\, RT\,(R - R_0)$$

If $n=2$, a vanishing fracture is involved. Finding the limiting value, we get—

$$\text{Volume} = 2\pi\, C \log_e \frac{R}{R_0}$$

If greater exactness is required, then we consider the trapezoidal sections which form a radial section of the machined disc. Let R_1, R_2 be the inner and outer radii of such a trapezoidal section, T_1, T_2, the corresponding thicknesses of the disc.

The area of the trapezoid is

$$(R_2 - R_1) \times \frac{T_1 + T_2}{2}$$

and the radius of the centroid is

$$R_1 + \frac{R_2 - R_1}{3} \times \frac{T_1 + 2T_2}{T_1 + T_2}$$

therefore, by Pappus' Rule—

$$\text{Volume} = 2\pi \left[R_1 + \frac{R_2 - R_1}{3} . \frac{T_1 + 2T_2}{T_1 + T_2} \right] \left[(R_2 - R_1)\; \frac{T_1 + T_2}{2} \right]$$

$$= \pi\,(R_2 - R_1) \left[R_1(T_1 + T_2) + (R_2 - R_1). \; \frac{T_1 + 2T_2}{3} \right]$$

The weights of the rim and boss have to be added to the weight of the disc proper to get the weight of the complete wheel.

(b) To find the index n when the shape of the disc is known,

we have—
$$T_1 R_1^n = T_2 R_2^n$$

therefore

$$\log T_1 + n \log R_1 = \log T_2 + n \log R_2$$

whence

$$n = \frac{\log T_1 - \log T_2}{\log R_2 - \log R_1}$$

(c) In these days we often have, in the case of marine geared turbines, to work out the critical speeds for torsional vibration. For this purpose we require to know the moments of inertia of the rotors. The moment of inertia of a disc of form tr^n = constant is

$$\int_{R_0}^{R} 2\pi r^3 t dr$$

$$= 2\pi C \int_{R_0}^{R} r^{3-n} dr$$

$$= 2\pi C . \frac{R^{4-n} - R_0^{4-n}}{4-n}$$

(d) In the disc of uniform strength we have

$$t = T_0 e^{-\gamma r^2}$$

$$\text{Volume} = \int_0^R 2\pi r T_0 e^{-\gamma r^2} dr$$

$$= \pi T_0 . \frac{1 - e^{-\gamma R^2}}{\gamma}$$

But $T_0\, e^{-\gamma R^2} = T$ the neck thickness. Therefore—

$$\text{Volume} = \pi \cdot \frac{T_0 - T}{\gamma}$$

(e) The moment of inertia of this disc is

$$\int_0^R 2\pi r^3 T_0\, e^{-\gamma r^2}\, dr$$

Integrating by parts, we obtain—

$$\pi \cdot \frac{T_0 - T - T\gamma R^2}{\gamma^2}$$

or, since $\gamma R^2 = \log_e \frac{T_0}{T}$, moment of inertia =

$$\frac{\pi}{\gamma^2}\left[T_0 - T\left(1 + \log_e \frac{T_0}{T}\right)\right]$$

30. MEMORANDA.

$e = 2.718281828$
$\log_{10} e = 0.4342945$
$g = 386$ inches per sec.^{2}

For steel, weight $= .283$ lb. per cub. inch.
For steel, $E = 30{,}000{,}000$ lbs. per sq. inch.
For steel, $\sigma = .25$

Centrifugal force in lbs. on weight of W lbs. at radius R inches and N revs. per min.
$= (\text{Mass}) \times (\text{Radius}) \times (\text{Angular Speed})^2$

$$= \frac{W}{g} \times R \times \left(\frac{2\pi N}{60}\right)^2$$

$$= W \times R \times \left(\frac{N}{187 \cdot 7}\right)^2$$

$$= 28.4 \times W \times R \times \left(\frac{N}{1000}\right)^2$$

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